Why TLM – Its Aims and Uses

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Any effective pedagogy for teaching mathematics to children must include materials, especially when new topics are introduced at the elementary level. There are three reasons for the above:

First, all concepts in mathematics are abstract. The concepts form in our minds slowly over time. As we play with concrete materials, our minds gradually grasp the general concepts related to numbers and shapes arising out of the concrete examples. So at the elementary level every new concept should be introduced with concrete material or manipulatives. We will give examples of manipulatives for teaching various aspects of whole numbers and how they link to algebra.

Second, mathematics includes a lot of conventions as well as rules, in particular in how we write numbers. Conventions, which have no logic as such, can only be learned through association.

Manipulatives help establish that association much more strongly than chalk-and-board or drawing because these things can be moved around. So they not only give visual input, but also tactile and kinaesthetic ones to the brain and thus help in creating associations. This is as true for learning place value and then using it for various operations as for algebra or geometry.

Third, math in particular tends to build up i.e. the concepts are organised in the hierarchical manner. So any lack of understanding at any level creates a big hurdle for further learning of concepts which depend on those at the previous level. For example, poor understanding of place value heavily impacts how well a child can (or rather cannot) master algorithms for addition, subtraction, multiplication and division with multi-digit numbers. The use of manipulatives builds a strong foundation that facilitates further learning. We will give one of many examples of one material that help decipher algorithms.

Example 1: The *ganitmala* [see Fig. 1.1], a mala of 100 beads in two contrasting colours alternating in groups of tens, is a great proportional material that makes such an association. This *mala* is basically a manipulative version of the non-negative number line. So beads are counted off from the left since the zero is to the left of all positive numbers in any number line. So when we show twenty five in the *ganitmala*, the twenty, or two tens, are on the left and the five, or five ones are on the right.

Figure 1.1

This is exactly how we write twenty five as

T	O
2	5

the 2 indicates how many tens, while the 5 is the count of the remaining ones. However, the order of writing tens on the left and ones on the right is only a convention. A curious child may ask why the tens are written on the left. The *ganitmala* demonstrates that the tens are on the left – an association that cannot be provided by bundles and sticks, which can be placed in any order.

The *ganitmala* can be used for introduction and comparison of numbers, as well as all four operations, so long as the numbers are within 100. It can also be used to find HCF. For integers, double *ganitmala* – with 200 beads in four colours can be used. Two more colours are needed for the negative part of the number line. This can be used to solve most equations with integer coefficients and with the variable (or unknown) appearing only once. Interested readers may look into the references below on double *ganitmala* for further details.

Example 2: Flats-Longs-Units (FLU) or 2D base-10 blocks:

These include bigger squares split into a 10 × 10 grid denoting hundred, 10 × 1 rectangles denoting tens and small squares denoting ones [see Fig 2.1]. The colour in all of these should be the same since the 10 pink 'ones' (units) should add up to a pink ten (long) and 10 pink tens should add up to a pink hundred (flat). If, for example, the colour is

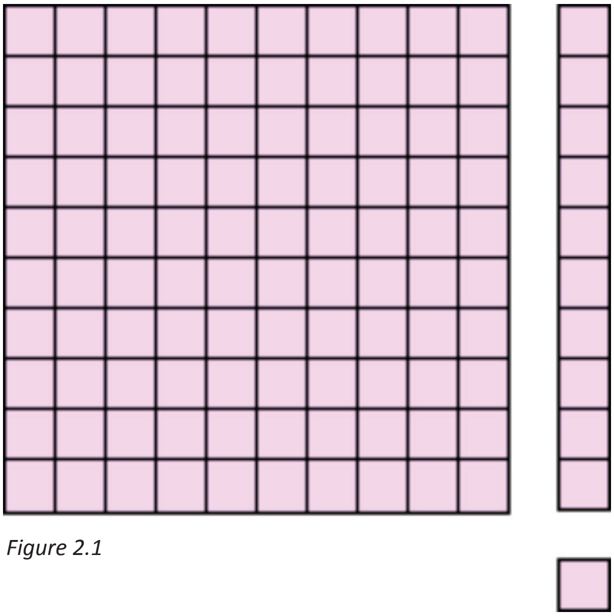

Figure 2.1

changed from pink to yellow that may be confusing for children. This is a very powerful teaching material that basically deciphers almost everything related to whole numbers (up to 3-digits) including algorithms in particular. It can even help finding the division algorithm for finding square roots!

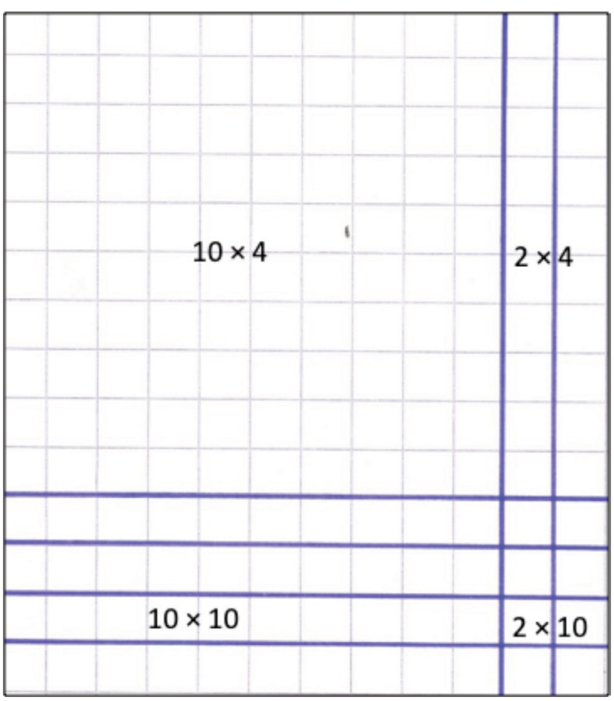

Figure 2.2

Incidentally, no one sells this material – at least not enough of it. To do all kinds of division problems within 3-digit numbers, each set should have at least 90 longs and 90 units. This takes care of most multiplication as well. We recommend 15-20 flats in each set. Sheets from square grid notebooks can be used for making this material. From each such sheet usually a hundred come off from one corner. The remaining reversed L shape should be harvested for longs and the corner of the L provides units. The layout for any such sheet is essentially a multiplication in disguise (see Fig. 2.2 representing 12 × 14).

This is completely in line with the grid multiplication included in several textbooks including NCERT.

The advantages of this layout, and FLU in general, carries over to decimals. Since children are older by then, graph papers are used along with colouring. For example:

In a centimetre graph paper, 10cm × 10cm square should be taken as 1.

So one-tenth of that, or 0.1, is a 1cm × 10cm rectangle.

Similarly 0.01 can be a 1cm × 1cm square or a thin 1mm × 10cm rectangle.

0.001 is a 1mm × 1cm rectangle while 0.0001 is a tiny 1mm × 1mm square.

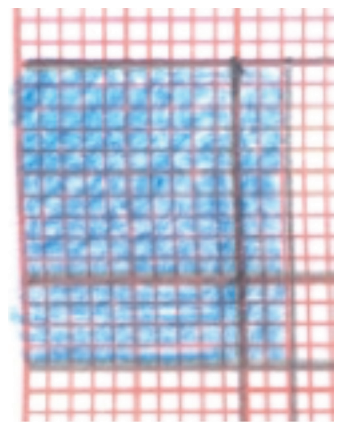

Figure 2.3

Now 0.12 × 0.14 becomes a miniature version of the earlier 12 × 14, thus making decimal multiplication a piece of cake.

The miniaturisation also aids in understanding that 0.12 × 0.14 is nothing but 12 × 14 divided by 100 × 100 which actually is a standard algorithm [see Fig. 2.3].

This stretches all the way into algebra. The FLU gets generalised into algebra tiles with two differences –

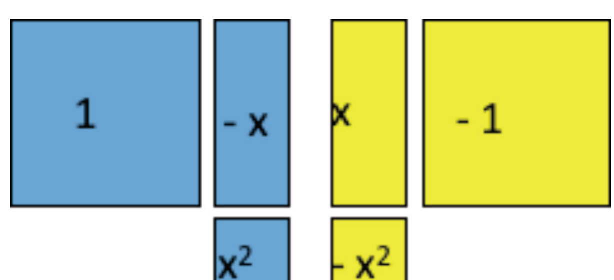

Figure 2.4

(i) The dimension of the rectangle (and therefore the ratio of the sides square) opens up from a fixed 1:10 and

(ii) Everything, i.e., big squares, small squares and rectangles are in two contrasting colours (which can be either double sided or two separate sets) representing positives and negatives [see Fig. 2.4].

(iii) The same layout continues for (x + 2)(x + 4) and with colour variation accounting for negativity for (x + 2)(x – 4), (x – 2)(x + 4) and (x – 2)(x – 4) [see Fig. 2.5]. In each i.e. whole numbers, decimals and algebra:

- The small squares are together – if bigger numbers like 32 × 54 or (3x ± 2)(5x ± 4) are used, the reader will be able to see that the big squares also stay together

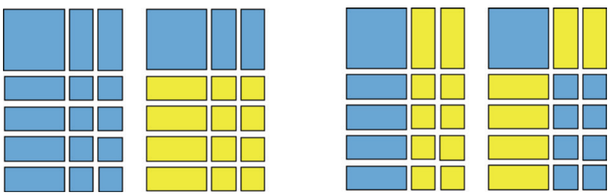

Figure 2.5

- The big squares and the small squares occupy opposite corners
- The rectangles are always in two parts in the remaining corners

A child familiar with FLU will naturally identify many such common patterns in algebra tiles as well. In fact, a lot of Vedic *ganit* can be explained with FLU. Certain notations can be understood better with FLU in two colours – positive and negative – like algebra tiles!

The best part of algebra tiles lies in understanding the colour patterns in the four examples shown above and using them to crack middle term factorisation. The reader can refer to the links at the end and is encouraged to explore further.

Example 3: Ten-frames with counters:

Ten-frames are 2 × 5 grids of 10 squares which are, used along with round counters. Ten-frames can be easily made from any card type materials including boxes, while buttons can be used for counters.

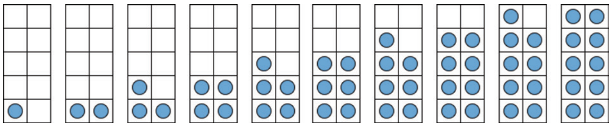

Figure 3.1

Numbers from 1-10 can be represented as follows using ten-frames and counters [see Fig. 3.1].

What jumps out is the alternating pattern – (i) an odd one jutting out at the top and (ii) level or even top. That is basically 'odd' and 'even'! Notice how this arrangement automatically connects the common use of 'odd' and 'even' to their mathematical meaning with respect to natural numbers. One can ask whether zero should be odd and it can be argued that since it does not have an odd counter jutting out it can't be odd. So it has to be even.

At this step, if an even number is added to another even number, then the top of the sum is level. If an odd number is added to an even number, then because of the odd number the top of the sum becomes non-even, that is, the total has an odd counter at the top, so it becomes odd. But if two odd numbers are added then the parts which jut out compensate each other by fitting together and we get an even number as a sum.

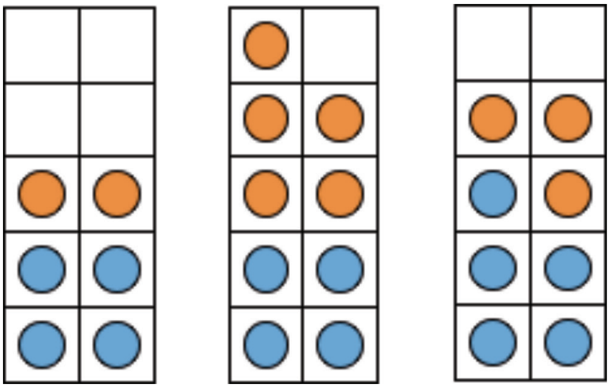

Figure 3.2

This visualisation of even and odd numbers [see Fig. 3.2] not only helps children understand the terminology, it also helps them figure out what happens when these numbers are added together.

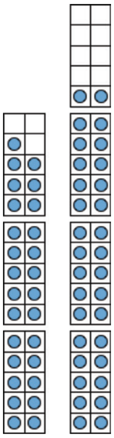

Figure 3.3

Again does zero behave the same way as the rest of the even numbers, that is to say, is the sum of zero plus an even number itself an even number? Can zero plus an odd number result in an odd number? This solidifies the case for zero being an even number.

The story does not end here!

The next step is understanding what happens with bigger numbers.

Before going into 3-digit numbers, a

child learns that any 2-digit number is made of tens and ones. So, using ten-frames, a number like 27 or 32 looks like this [see Fig. 3.3]. Since ten is an even number, several tens put together is still going to remain an even number. So the number of tens do not contribute to the oddness or evenness of a number: it is the number of ones, that is, the unit's digit determines if a number is odd or even. Since hundreds are made of tens and the same applies to thousands as well as all higher place-value units like lakhs and millions, the same technique works for any natural number, no matter how many digits it has. In all cases, we can just look at the unit's digit and figure out whether it is odd or even.

Is the above different from a more formal definition: An even number is a number that is divisible by 2, or one that can be written as 2n for some n = 0, 1, 2, 3, 4 …? Or that an odd number is any number that is not divisible by 2, which leaves a remainder of 1 when divided by 2 or that can be written as 2n + 1 for some n = 0, 1, 2, 3, 4…?

The reader could think, well isn't n = 3 for odd number 7 and even number 6? What do you think n is for 11 or for 26? In light of the visualisation mentioned above, n is nothing but the common height of both columns. It is also the quotient of the number divided by 2 while the number which is jutting out is the remainder, which of course would be missing for even numbers.

Also the algebraic proof of odd number + odd number = even illustrated as $(2m + 1) + (2n + 1) = 2(m + n) + 1 + 1 = 2(m + n + 1)$ is clearly illustrated by the ten-frame visuals. For any child familiar with the ten-frame visuals, it becomes so much clearer and meaningful. Whereas otherwise it can be just some algebraic jugglery with symbols which may be too scary to touch. The reader is encouraged to make similar connections for even number + odd number and even number + even number.

Ten-frames have other uses as well and we strongly encourage the reader to explore them! They are particularly helpful for various automatisations which aid learning.

Up to Class 10, mathematics is a compulsory subject for every child regardless of his/her inclination and/or proficiency in it. Teaching materials as aids help most children understand better and while giving a lot of food for thought for the mathematically-oriented. Hence most textbooks nowadays, including the NCERT ones, mention a whole range of materials. However, teachers are often not so familiar with them and may not have seen them outside textbooks. So much remains unexplored and not utilised.

References:

- Dhankar, R: Teaching and Learning of Mathematics https://azimpremjiuniversity.edu.in/SitePages/pdf/The-Teaching-and-Learning-of Mathematics.pdf
- *Ganitmala* – a visual manual: http://teachersofindia.org/en/activity/visual-manual-ganitmala-0#
- Double *ganitmala*:
  - Intro to integers: http://www.teachersofindia.org/en/presentation/integers-ganit-mala
  - Comparing integers: http://www.teachersofindia.org/en/presentation/comparing-integers-ganit-mala
  - Adding integers: http://www.teachersofindia.org/en/presentation/addition-ganit-mala
  - Subtracting integers: http://www.teachersofindia.org/en/presentation/subtraction-ganit-mala
  - Initiating equations: http://teachersofindia.org/en/presentation/solving-equations-ganit-mala
  - Simple equations: http://teachersofindia.org/en/presentation/solving-simple-equations-ganit-mala
  - Complex equations: http://teachersofindia.org/en/presentation/solving-complex-equations-ganit-mala
- FLU in multiplication: http://teachersofindia.org/en/presentation/initiating-multiplication
- Algebra tiles:
  - Demo: http://www.youtube.com/watch?v=4AwXOibqGxI
  - Practice: http://illuminations.nctm.org/Activity.aspx?id=3482

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